

With an Eye on the Mathematical Horizon: Dilemmas of Teaching Elementary School Mathematics

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Abstract

Ideas like "understanding," "authenticity," and "community" are central in current debates about curriculum, instruction, and assessment. Many believe that teaching and learning would be improved if classrooms were organized to engage students in authentic tasks, guided by teachers with deep disciplinary understandings. Students would conjecture, experiment, and make arguments; they would frame and solve problems; and they would read, write, and create things that mattered to them. This article examines the challenge of creating classroom practices in the spirit of these ideals. With a window on her own teaching of elementary school mathematics, the author presents three dilemmas—of content, discourse, and community—that arise in trying to teach in ways that are, in Bruner's terms, "intellectually honest." These dilemmas arise reasonably from competing and worthwhile aims and from the uncertainties inherent in striving to attain them. The article traces and explores the author's framing of and response to these dilemmas, providing a view of the pedagogical complexities that underlie current educational visions and the conditions needed to work toward them.

Over the past few years, school mathematics has been the target of a wave of reform recommendations (California State Department of Education, 1985; National Council of Teachers of Mathematics [NCTM], 1989, 1991; National Research Council, 1989, 1990). Rather than acquiring basic operations and terminology, students should be "doing mathematics" (NCTM, 1989, p. 7). Students should learn to look for patterns and frame problems (National Research Council, 1990); to "explore, conjecture, and reason logically" (NCTM, 1989, p. 5); and to engage in mathematical argument within a community in which standards of math-

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ematical evidence form the basis for judging correctness (NCTM, 1991).

These ideas about reforming school mathematics are part of a broader set of contemporary ideas about improving education. Across school subjects, current proposals for educational improvement are replete with notions of "understanding," "authenticity," and "community"—about building bridges between the experiences of the child and the knowledge of the expert. Teaching and learning would be improved, so the argument goes, if classrooms were organized to engage students in authentic tasks, guided by teachers with deep disciplinary understandings. Students would conjecture, experiment, and make arguments; they would frame and solve problems; and they would read, write, and create things that mattered to them. Teachers would guide and extend students' intellectual and practical forays, helping them to extend their ways of thinking and what they know as they develop disciplined ways of thinking and encounter others' texts and ideas.

These ideas are not new. Our contemporary hopes are rooted in the visions of our educational forebears—among them thinkers such as Bruner, Dewey, and Schwab. The notion that school curriculum could be much more substantial underlies Schwab's (1961/1974) claim that the disciplines themselves hold images of what learning entails. Examining what writers do, what it means to "know" something in history, how ideas develop in scientific communities—each of these, according to Schwab, should inform what students do in school. Dewey (1902) argued that we should think much more fluidly about the links between the lives and minds of children and the notion of "knowledge." We should, he claimed,

abandon the notion of subject-matter as something fixed and ready-made in itself, outside the child's experience; cease thinking of the child's experience as also something hard and fast; see it as some-

thing fluent, embryonic, vital; and we realize that the child and the curriculum are simply two limits that define a single process. Just as two points define a straight line, so the present standpoint of the child and the facts and truths of studies define instruction. It is continuous reconstruction, moving from the child's present experience out into that represented by the organized bodies of truth that we call studies. [Dewey, 1902, p. 11]

Both Schwab and Dewey press me to think hard about my conception of curriculum and of the ways in which children might encounter content. Bruner's (1960) notion of "intellectual honesty," however, has most captured my imagination. Writing on the topic of readiness for learning, he argued: "We begin with the hypothesis that any subject can be taught effectively in *some intellectually honest form* to any child at any stage of development. It is a bold hypothesis and an essential one in thinking about the nature of a curriculum. No evidence exists to contradict it; considerable evidence is being amassed that supports it" (Bruner, 1960, p. 33, emphasis added).

Any subject? To *any* child? At *any* time? My undergraduate students sometimes squirm a bit and make weak, nervous, jokes. "Calculus? Can a first grader learn calculus?" But I, more experienced with young children, am quite convinced. The things that children wonder about, think, and invent are deep and tough. Learning to hear them is, I think, at the heart of being a teacher. David Hawkins's (1972) wonderful essays provide multiple illustrations of the insights of young children and of the special dispositions entailed in the capacity to recognize such insights.

Bruner has with this passage inspired me to rethink what would be worthwhile activity for my third graders. The idea of intellectual honesty makes sense. Somehow what I do with children should be honest, both to who they are and to what I am responsible to help them learn. Intellectual honesty implies twin imperatives of responsiveness and responsibility. But I won-

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der: *How* do I create experiences for my students that connect with what they now know and care about but that also transcend their present? *How* do I value their interests and also connect them to ideas and traditions growing out of centuries of mathematical exploration and invention?

This article is about my investigation of these issues. Using myself as the object and tool of my inquiry, I teach mathematics daily to a heterogeneous class of third graders at a local public elementary school. Over half of the students are from other countries and speak limited English; the American students are diverse ethnically, racially, and socioeconomically and come from many parts of the United States. Sylvia Rundquist, the teacher in whose classroom I work, teaches all the other subjects besides mathematics. She and I meet regularly to discuss individual students, the group, what each of us is trying to do, and the connections and contrasts between our practices. We also spend considerable time discussing mathematical ideas, analyzing representations generated by the students or introduced by me, assessing the roles played by me and by the students in the class discussions, and examining the children's learning. My goal in this work is not to make claims about what other teachers should or should not do. Rather, my aim is to investigate some of the issues that arise in trying to teach mathematics in the spirit of the current reforms. It is a kind of research *into* teaching complementary to other research on teaching. By doing this teaching myself, I can offer a perspective that is different from—not better than—what can be learned from other vantage points or methods.

This article draws on data from my teaching during 1989–1990. In this particular class, in which we had 22 students, 10 were from the United States, and 12 were from other countries—Indonesia, Taiwan, Korea, Nepal, Nigeria, Kenya, Egypt, Ethiopia, Nicaragua, and Canada. Four of the 10 U.S. students were African American. Although no standardized or district testing is

done in this school until the end of third grade, informal assessment showed that these students' entering levels of mathematics achievement varied widely in both mathematical skills and concepts.

The mathematics period in my class is approximately 1 hour long. During this time, we often work on just one or two problems. My intention is to select problems that will be generative, rich with mathematical possibility and opportunity. Usually the problems are built from the previous day's work, with an eye to where we need to head. The class often begins with students exploring the problem of the day individually. As part of the class, students keep mathematics notebooks in which they record all their work and in which they also write about that work. Although I ask students to start by spending some time thinking about and working on the problem alone, I also encourage them to confer with others sitting nearby. After 10 minutes or so (depending on the problem), students move into small groups and work further together. We spend about half the class period in a whole-class discussion, during which individuals and groups present their solutions and discuss the ideas embedded in the problems. The character of some of these discussions is illustrated in this article.

Every class period is audiotaped, and most are videotaped as well. I keep a daily journal about my thinking and work. I also give quizzes and homework. The notebooks, quizzes, and homework are all photocopied and saved. To complement what can be learned from their written work, students are interviewed regularly, sometimes informally, sometimes more formally, sometimes in small groups and sometimes alone. Semistructured interviews at the beginning and end of the year explore the students' ideas and feelings about school, their developing epistemological orientations and beliefs, as well as their understanding of a sample of mathematical topics (e.g., place value and regrouping, fractions, in-

tegers, polygons, and probability). Informal interviews across the year probe students' reactions to classroom events and their developing ideas about particular content. I have also been developing ways to conduct whole-class interviews on a regular basis.

Among my aims is that of developing a practice that respects the integrity both of mathematics as a discipline *and* of children as mathematical thinkers. Three components of mathematical practice frame my work: the content, the discourse, and the community in which content and discourse are intertwined. Students must learn mathematical language and ideas that are currently accepted. They must develop a sense for mathematical questions and activity. They must also learn how to reason mathematically, including an understanding of the role of stipulation and definition, of representation, and of the difference between illustration and proof (Kitcher, 1984; Putnam, Lampert, & Peterson, 1990). Schoenfeld (1989, p. 9) argues: "Learning to think mathematically means (a) developing a mathematical point of view—valuing the processes of mathematization and abstraction and having the predilection to apply them, and (b) developing competence with the tools of the trade, and using those tools in the service of the goal of understanding structure—mathematical sense-making."

Because mathematical knowledge is socially constructed and validated, sense making is both individual and consensual. Drawing mathematically reasonable conclusions involves the capacity to make mathematically sound arguments to convince oneself and others of the plausibility of a conjecture or solution (Lampert, in press). It also entails the capacity to appraise and react to others' reasoning and to be willing to change one's mind for good reasons. Thus, community is a crucial part of making connections between mathematical and pedagogical practice.

I take a stance of inquiry toward my practice, working on the basis of conjectures about students and understandings of the

mathematics; in so doing, both my practice and my understandings develop. In the service of helping 8- and 9-year-old children learn, I seek to draw on the discipline of mathematics at its best. In so doing, I necessarily make choices about where and how to build which links and on what aspects of mathematics to rest my practice as a teacher. With my ears to the ground, listening to my students, my eyes are focused on the mathematical horizon. In this article, I explore the tensions I experience as I face this challenge.

A Restatement of the Pedagogical Challenge

Bruner (1960) argues that children should encounter "rudimentary versions" of the subject matter that can be refined as they move through school. This position, he acknowledges, is predicated on the assumption that "there is a continuity between what a scholar does on the forefront of his discipline and what a child does in approaching it for the first time" (pp. 27–28). Schwab (1964/1971), similarly, outlines a vision of the school curriculum "in which there is, from the start, a representation of the discipline" (p. 269) in which students have progressively more intensive encounters with the inquiry and ideas of the discipline. But what constitutes a defensible and effective "rudimentary version"? And what distinguishes intellectually honest "fragments of the narrative of enquiry" (Schwab, 1961/1974) from distortions of the subject matter?

Wineburg (1989) argues that school subjects have strayed too far from their disciplinary referents. I agree. Still, trying to relate them to the disciplines is neither straightforward nor without serious conceptual and philosophical problems (Palincsar, 1989). Before considering the dilemmas that emerge in my own efforts to teach third-grade mathematics, I note three problems inherent in attempting to model classrooms on ideas about authentic mathemat-

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ical practice, problems that persuade me to avoid the term "authentic" in this context.

First, constructing a classroom pedagogy on the discipline of mathematics would be in some ways inappropriate, even irresponsible. Mathematicians focus on a small range of problems, working out their ideas largely alone. Teachers, in contrast, are charged with helping *all* students learn mathematics, in the same room at the same time. The required curriculum must be covered and skills developed. With 180 days to spend and a lot of content to visit, teachers cannot afford to allow students to spend months developing one idea or learning to solve a certain class of problems. And the best and seemingly most talented must not be alone in developing mathematical understanding and insight. Moreover, certain aspects of the discipline would be unattractive to replicate in mathematics classrooms. For instance, the competitiveness among research mathematicians—competitiveness for individual recognition, for resources, and for prestige—is hardly a desirable model for an elementary classroom. Neither is the aggressive, often disrespectful, style of argument on which much intradisciplinary controversy rests (Boring, 1929). Finally, in any case, modeling classroom practice on *the* discipline of mathematics is, of course, impossible. As Schwab (1964/1971) points out, disciplines have multiple structures; these structures are also not easily uncovered. No one "knows" the structures of mathematics; there is no single view of "what mathematics is." My work, therefore, aims to create and explore practice that tries to be intellectually honest to both mathematics and the child. In this article I present and analyze three dilemmas I encounter in trying to create a practice of mathematics teaching that is defensibly—but not solely—grounded in mathematics.

The three dilemmas arise out of the contradictions inherent in weaving together respect for mathematics with respect for students in the context of the multiple purposes of schooling and the teacher's

role. Teachers are responsible for helping each student learn particular ideas and procedures, accepted tools of mathematical thought and practice. However, a view of mathematics that centers on learning to think mathematically suggests that the teacher should not necessarily show and tell students how to "do it" but that they should instead learn to grapple with difficult ideas and problems. Yet creating such learning experiences may result in frustration and surrender rather than confidence and competence. Fostering a classroom mathematical community in the image of disciplinary practice may lead students to become confused—or to invent their own, nonstandard, mathematics. The teacher thus faces contradictory goals. As Lampert (1985, p. 181) writes, "The juxtaposition of responsibilities that make up the teacher's job leads to conceptual paradoxes" with which the teacher must grapple, and for which there are not single "right" choices. This is because the teacher "brings many contradictory aims to each instance of her work, and the resolution of their dissonance cannot be neat or simple" (Lampert, 1985, p. 181). In trying to teach mathematics in ways that are intellectually honest—to the content and to students—I find myself frequently facing thorny dilemmas of practice. I explore three such dilemmas here. Rooted in the three components of mathematical practice that frame my work, one dilemma centers on representing the content, another on respecting children as mathematical thinkers, and the third on creating and using community. In each case, I begin by framing the dilemma. I have selected one example from my classroom to illustrate each dilemma, and I use a common structure across these examples. First, because the dilemmas arise directly from my explicit goals, I give some rationale for what the students and I were doing in the example. Why were we spending time on negative numbers in third grade? Why do I think it valuable for students to experiment with and invent math-

ematical ideas? What do I mean by "community," and why is it important to create and use such a community in helping third graders learn mathematics? Next, I describe and explain what I faced in each case and why it presented a dilemma for me. To elaborate on the presentation of the illustrative cases, I provide evidence of students' thinking and change. I conclude each case with a discussion of the dilemmas entailed.

Dilemma No. 1: Representing the Content: The Case of Teaching Negative Numbers

What concerns [the teacher] is the ways in which the subject may become part of experience; what there is in the child's present that is usable in reference to it; [to] determine the medium in which the child should be placed in order that growth might be properly directed. [Dewey, 1902, p. 23]

How can 9-year-olds be engaged in exploring measurement, addition and subtraction, fractions, and probability? What are the hooks that connect the child's world with particular mathematical ideas and ways of thinking? Shulman and his colleagues (e.g., Shulman, 1986, 1987; Wilson, Shulman, & Richert, 1987) have charted brave new territory with the concept of *pedagogical content knowledge*, "the most powerful analogies, illustrations, examples, explanations, and demonstrations—in a word, the ways of formulating and representing the subject that make it comprehensible to others" (Shulman, 1986, p. 9). Shulman (1986) argues that a teacher must have "a veritable armamentarium of alternative forms of representation" (p. 9); moreover, the teacher must be able to "transform" his or her personal understandings of the content. In Dewey's terms, "to see it is to psychologize it." Figuring out powerful and effective ways to represent particular ideas implies, in balanced measure, serious attention to both the mathematics and the children. This is more easily said than done. I

will illustrate this with an account of my struggles to find a way of helping my third graders extend their domain from the natural numbers to the integers.

Rationale for Teaching Integers

When there is so much content to cover, why is teaching third graders about negative numbers worthwhile or appropriate? The justifications, I contend, are both experiential and mathematical. Children who live in Michigan know that there are a few days every winter when the temperature is below zero—and that means that it is too cold to go outside for recess. Many have also had experience with owing someone something or being "in the hole" in scoring a game—conceptually, experiences with negative numbers that have not been symbolically quantified. Still, they assert with characteristic 8-year-old certainty that "you can't take 9 away from 0." That third graders—indeed, many older children—think the "lowest number" is zero seems problematic. Teaching them about negative numbers is an attempt to bridge their everyday quantitative understandings with formal mathematical ones.

Analyzing the Content and Thinking about Learners

I began my pedagogical deliberations by reviewing the various models for negative numbers that I knew. The school district's curriculum (*Comprehensive School Mathematics Program* [CEMREL, 1979]) uses a story about an elephant named Eli who has both regular and magic peanuts. (The *Comprehensive School Mathematics Program* [CSMP] is an innovative mathematics curriculum developed in the 1970s for elementary classrooms [see Remillard, 1990, for a comparison of CSMP with other elementary mathematics programs].) Whenever a magic peanut and a regular peanut are in Eli's pocket at the same time, they both disappear (i.e., $-1 + 1 = 0$). This representation did not appeal to me, although I was sure that it would be fun and

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engage the children. I was concerned about the messages entailed in fostering “magical” notions about mathematics—peanuts just disappearing, for example—because of the widespread tendency to view mathematics as mysterious and beyond sense or reason. After considering other models such as money (and debt), a frog on a number line, and game scoring, I decided to use a building with many floors both above and below ground (see Figure 1). As is the case in many other countries, the ground floor is called the “0th” floor. This was a considerable adaptation of a model called the “Empire State Building” that appears briefly for one lesson in CSMP.

Why did I settle on this admittedly fantastic model? Analyzing negative numbers and operations with them, I saw that there were at least two important dimensions:

Negative numbers can be used to represent an *amount of the opposite of something* (e.g., -5 can represent a \$5 debt, the opposite of money).

Negative numbers can be used to represent a *location relative to zero* (e.g., -5 can represent a position that is five units away from zero).

Any number has two components: *magnitude* and *direction*; from a pedagogical point of view, this seems to become particularly significant when the students’ domain is stretched to include negative numbers. A focus on the magnitude component leads to a focus on *absolute value*. This component emerges prominently in many everyday uses of negative numbers (e.g., debt, temperature). Thus, comparing magnitudes becomes complicated. There is a sense in which -5 is more than -1 and equal to 5 , even though, conventionally, the “right” answer is that -5 is less than both -1 and 5 . This interpretation arises from perceiving -5 and 5 as both five units away from zero and -5 as more units away from zero than -1 . Simultaneously understanding that -5 is, in one sense, more than -1 and, in another sense, less than -1 is at the heart of understanding negative numbers.

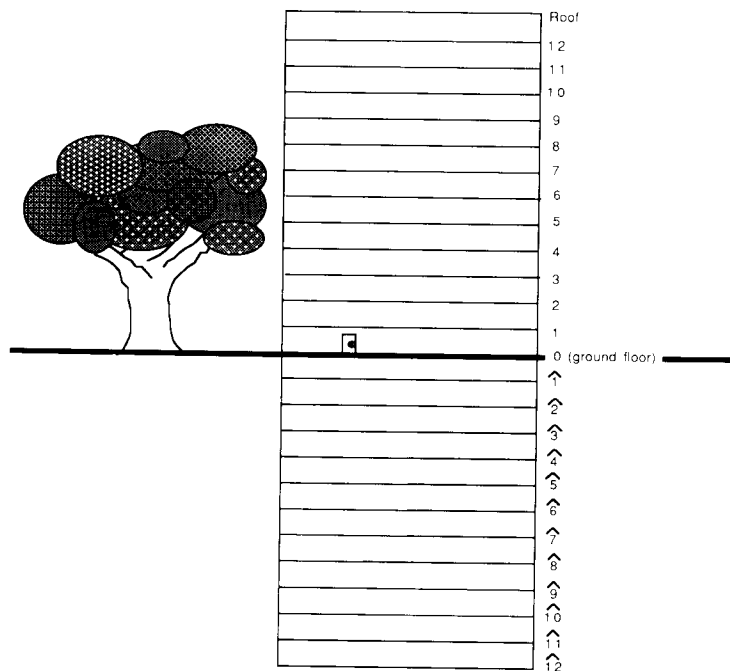


FIG. 1.—The “building”: A model for adding and subtracting integers

Just before beginning to work with negative numbers, I wrote in my journal:

I'm going to try the elevator model because its advantages seem to outweigh its disadvantages. It's like the number line in that it, too, is a positional model. The "up" and "down" seem to make sense with addition and subtraction: Artificial rules don't have to be made. And I think I might be able to use it to model adding and subtracting negative numbers as well as positives: When a person wants to add more underground floors, that would be *adding* negatives. If someone wants to demolish some of the underground floors, that would be *subtracting negatives*. But could something like $4 - (-3)$ be represented with this model? I'm not sure. I guess it would be like *moving away from the underground floors*, but . . . I don't like the money or game models right now because they both seem to fail to challenge kids' tendency to believe that negatives are the same as zero (owing someone five dollars—i.e., -5 —seems the same as having no money). [Teaching journal, September 25, 1989, p. 28]

In this journal entry, I was settling on the building representation after weighing concerns for the essence of the content, coupled with what I knew to expect of 8-year-olds' thinking—for instance, that they tend to conceive negative numbers as just equivalent to zero. I hoped that this clearly positional model would help to deflect that tendency. I was aware from the start that the model had mathematical limits—for example, its capacity to model the subtraction of negative numbers.

We began to work with the building by labeling its floors. I was pleased to see that the students readily labeled the underground floors correctly. I used the language implied by the building: we had floors below the ground, sometimes referred to as "below zero." The circumflex (\wedge) above the numerals on the building pictured in Figure 1 replaced the traditional negative sign ($-$). This is a convention from CSMP. The rationale for substituting the circumflex for

the minus sign is to focus children on the idea of a negative number as a *number*, not as an *operation* (i.e., subtraction) on a positive number. Although I have found the symbol useful pedagogically, I do not use it in this article because it would be unfamiliar to most readers. We had floors above the ground, sometimes referred to as "regular floors." The unconventional system—with zero as the ground floor—did not seem to confuse the students who were, as a group, relatively unfamiliar with multistory buildings of any kind. I introduced little paper people who rode the elevator in the building: "Take your person and put her on any floor. Have her take the elevator to another floor and then write a number sentence to record the trip she took." Thus, if a person started on the fourth floor and came down six floors, we would record $4 - 6 = -2$. If a person got on at the second floor below ground and rode up five floors, this would be written as $-2 + 5 = 3$. I introduced these conventions of recording because I wanted to convey that mathematical symbols are a powerful way of communicating ideas, a consistent theme in my goals.

We worked on increasingly complicated problems with the building, for example, "How many ways are there for a person to get to the second floor?" This problem generated an intense discussion. Some children negotiated long, "many-stop" trips for their little paper people, for example, $-5 + 10 - 6 - 4 + 3 + 6 - 2 = 2$. Others stuck with "one-stop" trips, for example, $-3 + 5 = 2$. The students debated: Were there infinite solutions? Or exactly 25 solutions? This argument afforded us the opportunity to talk about the role of assumptions in framing and solving problems. Those who assumed one-stop trips were right when they argued that there were exactly 25 solutions to this problem. Our arguments about this evolved quickly from one child's proposition that there were 24 solutions—she argued that there were 12 floors above and 12 floors below zero—to another child's observation that the ground floor offered one

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more solution: $0 + 2 = 2$. However, those who assumed that trips could be as long as you like were also right when they argued that the problem might have “afinidy” or maybe 8,000,000,000,000,000,000,000 solutions. As usual, the third graders reached out to touch the notion of the infinite with great fascination—and “afinidy” and 8,000,000,000,000,000,000,000 are virtually equivalent when you are 8 or 9.

The work with the building generated other wonderful explorations. Nathan noticed that “any number below zero plus that same number above zero equals zero,” and the children worked to prove that his conjecture would be true for all numbers.¹ Ofala produced “any number take away double that number would equal that same number, only below zero” (e.g., $5 - 10 = -5$).

Despite much good mathematical activity, such as our discussions of the number of solutions and the conjectures of Nathan and Ofala, I worried about what the students were learning about negative numbers and about operations with them. Writing the number sentences seemed somewhat perfunctory. I was not convinced that recording the paper people’s trips on the elevator was necessarily connecting with the children’s understandings of what it means to add or subtract with integers. I also saw that only partial meanings for addition and subtraction were possible with this model. For addition, we were only able to work with a change model (i.e., you start on the third floor and you go up two floors—your position has *changed* by two floors). For subtraction, we could model its comparison sense, but not the sense in which subtraction is about “taking away.” I also thought that the building was not helping students develop a sense that -5 was less than -2 . Although being on the fifth floor below ground was lower than the second floor below ground, it was not necessarily less. I wondered, not for the first or last time, about what may be gained by using and “milking” one representation thor-

oughly versus what may be gained by introducing multiple representations.

Finally, we hit a crisis—over what to do with $6 + (-6)$ (October 12, 1989). There was no sensible way to deal with this on the building. “If a person began at the sixth floor above the ground, what would it mean to go up ‘6 below-zero floors’?” The children struggled with trying to make sense:

- Betsy: Here’s how I do it. (She put a person on the sixth floor and on the sixth floor below ground and moved them toward each other.) 1 . . . 2 . . . 3 . . . 4 . . . 5 . . . 6. And so I move them both at the same time. And I got 0.
- Sean: But it says *plus*, not minus!
- Betsy: But you’re minusing!
- Riba: (to Betsy) Where’d you get the minus?
- Sean: You should just leave it alone. You can’t *add* six below zero, so you just leave it. Just say “good-bye” and leave it alone and it is still just six.
- Mei: But this six below zero would just disappear into thin air!
- Sean: I know. It would just disappear because it wouldn’t be able to *do* anything. It just stays the same, it stays on the same number. Nothing is happening.

Trying Money

After some deliberation, I decided to try money as a second representational context for exploring negative numbers. Money had some advantages that the building lacked: it was not positional and it seemed as though it would work better for modeling relative quantities—that -5 was less than -2 , despite the fact that 5 was more than 2. Moreover, all meanings for addition and subtraction were possible. The expression $6 + -6$ could have meaning: having 6 dollars and also owing 6 dollars. Still, I saw potential problems on the horizon. As I wrote in my journal, “One keeps bumping into the absolute value aspect of negative numbers—for example, $-\$5$ ($\$5$ of debt is

more debt than $-\$2$. You have to talk about how much *money* (or 'net worth') in order to make it focus on negative numbers being *less*" (Teaching journal, October 12, 1989, p. 63).

I struggled with the language "props" that would structure the fruitful use of money as a representation for negative numbers. I decided that I needed an 8-year-old's version of "net worth" so as to focus the children on the inverse relationship between debt and money, on financial *state* rather than on *actions* of spending or getting money. Our first money problem—about their teacher, Mrs. Rundquist—was very structured as I tried to create the representational context (see Ball, in press). Instead of our usual pattern of some small-group or independent work followed by a whole-class discussion, we discussed this problem together:

Ms. Suzuka has 4¢ (represented two ways:
"4¢" and with 4 magnetic checkers).

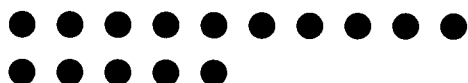


She wants to buy a pencil that costs 10¢.
(She needed _____?)

She borrowed 6¢ from Mrs. Rundquist.
Mrs. Rundquist gave her an IOU for 6¢
(written as " $-6¢$ " and also represented
with six "negative" checkers).



Later, she was lucky and got an envelope
with 15¢ in it.



She had to pay back Mrs. Rundquist.
What did she have for herself then?

To resolve the debt, I had the students pay off 1¢ of what was owed at a time, matching one negative checker (representing 1¢ of debt) with one regular checker (representing 1¢). We arrived at the answer of 9¢

without much difficulty. When I wrote the 6¢ debt as -6 , I asked, "Why do you think I wrote 6 below zero?" It seemed that the students found it sensible.

But when I asked them to write a number sentence to represent the story, they wrote $15 - 6 = 9$, not using any negative numbers at all. And that made sense when I thought about it: Their number sentences represented the *action* of paying off the debt (i.e., you take 6¢ from your 15¢ and give it to the person you owe). I realized that I would need to structure the use of this representation to focus on *how much money there was*, rather than on *actions*. I decided to do another problem in which the key question would be how much money Ms. Suzuka had *for herself* at any given point. So when she owes Mrs. Rundquist \$10 and also has \$13 in her pocket, one can ask "What does Ms. Suzuka have *for herself* right now?" and that would support the use of negative numbers—that is, $-10 + 13 = ?$ I conceived the idea of "for herself" as a representation of net worth; I hoped it would focus the students on balancing debt with money in ways that would illuminate positive and negative numbers. I felt that, if I could get it to work, money would be a good complement to our work with the building.

I realized as we continued, though, that the students did not necessarily reconcile debt with actual money, that they were inclined to remember both but to keep them separate. For example, if I talked about Jeannie having \$4 in her pocket and owing \$6 to her mother, they were not at all disposed to represent her financial state (how much money she had for herself) as $-\$2$. Instead, they would report that "Jeannie has \$4 and she *also* owes her mother \$6." With money, they seemed to avoid using negative numbers—maybe precisely because the representation entails quantity, not position. As Jeannie argued, quite rightly, "There is no such thing as below-zero dollars!"

Negative numbers seemed sensible on the building to denote different positions rel-

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ative to the ground. But, on the building, we used negative numbers only in the first or answer positions of the number sentences, $___ + 2 = ___$, because we never figured out what it would mean to move a negative amount on the building. With money, many children never used negative numbers to represent debt: They were inclined to report that someone had "\$6" and "also owes so-and-so eight dollars" rather than using $-\$8$ to represent the debt. They were also inclined to leave positive values (money) and negative ones (debt) unresolved.

Students' Learning

Uncertain about where we were and where we could reach, I gave a quiz. I found that, after exploring this new domain via the representations of the building and money, all the students were able to compare integers correctly, for example,

$$\begin{array}{l} -35 < 6 \\ 6 > -6 \end{array}$$

and explain why (e.g., " -35 is below zero and 6 is above zero so -35 is less than 6 "). They were also all newly aware that there is no "smallest number." However, about half of them, when asked for a number that was less than -4 , produced one that was more (e.g., -2). Note that producing a number less than -4 requires still more solid understanding of negative numbers than comparing a negative with a positive number. It is easier to see that -35 is less than 6 than to see that it is less than -6 . Children will typically explain that -35 is less than 6 simply because it is below zero and 6 is not. But when they examine -35 and -6 , they are often inclined to think that -35 is greater than -6 , on the basis of the magnitude of the numbers. When students focused only on the magnitude of the number, -2 seemed less than -4 . As I thought about how wary some of them still were of "these numbers" below zero, I reminded myself that it took over a thousand years for negative numbers to be accepted in the mathematical community—due principally

to their fundamental "lack of intuitive support" (Kline, 1970, p. 267). Why should I expect my third graders to be quicker to accept a difficult idea?

Dilemmas of Content and Representation

Clearly, the representation of negative numbers is fraught with dilemmas. I had to think hard about numbers below zero. And as I did so, I realized how rare such content analyses are for any of the topics typically taught. Moreover, the children's understandings and confusions provided me with more information with which to adapt my choices, yet the mathematics helped me to listen to what they were saying. Thus, it was in the ongoing weaving of children and mathematics that I constructed and adapted my instruction.

My analysis made me aware of how powerful the absolute value aspect of integers is—that is, that -5 is in many ways more than 2: It is farther from zero than 2 is. Moreover, -5 is also equal to 5 in some senses: they are equidistant from zero. So, given this insight, I faced the dilemma of what I should try to get my students to learn: could they learn to manage simultaneously the sense in which -5 is more than 3 and the sense in which it is less than 3? In school, they will be required to say that -5 is less than 3. Am I confusing them when I allow them to explore multiple dimensions of negative numbers and what these numbers represent?

I also had to think about what 8-year-olds could stretch to understand. Although there is research on student thinking, it has not investigated many topics that teachers teach: what are 8-year-olds' conceptions of proof and of what makes something true—in different domains? Our knowledge about primary-grade children's solving of arithmetic word problems or of place value, for instance, does not necessarily help us to predict how they understand the notion of numbers below zero, or the relationship between positive and negative integers.

Constructing good instructional representations and figuring out how to use them well are not the same thing. Even after I developed these two models for negative numbers and had analyzed them with respect to the mathematics and to each representation's accessibility for students, I still had to figure out how to use them—what kinds of problems to work on while using each tool, what should be the supporting language that would structure and focus the representation's key features for illuminating the content. For example, I discovered that I needed some kind of notion of "net worth" in order to steer the children's use of the money model away from attention to *actions*—buying (subtraction) or earning (adding)—to attention to *balances* and states. If I wanted the students to have a need to use negative numbers to represent quantities, then how money was engaged as a representational context was crucial (Ball, in press).

No representations capture all aspects of an idea, nor are all equally useful for particular students. No formulas exist for generating fruitful representations. Good teachers must have the capacity or be provided with the support to probe and analyze the content so that they can select and use representations that illuminate critical dimensions of that content for their students. Threaded throughout must be thoughtful consideration of students' current ideas and interests. A teacher must also figure out how to support and use the representational contexts that students construct. And teachers need alternative models to compensate for the imperfections and distortions in any given representation (Ball, 1988). When Bruner (1960) argues that constructing intellectually honest representations "requires a combination of deep understanding and patient honesty to present physical or other phenomena in a way that is simultaneously exciting, correct, and rewardingly comprehensible" (p. 22), this is no simple observation. As I try to do this, I struggle with dilemmas and unanswered questions.

And, if all the uncertainties were not enough, I face persistent uncertainties about what sense my students are making, about what they are learning.

Dilemma No. 2: Respecting Children as Mathematical Thinkers: The Case of "Sean Numbers"

Good teachers respect children's thinking. They view students as capable of thinking about big and complicated ideas, although what that actually means in mathematics is, at times, not clear. Mathematics is, after all, a domain in which there *are* "right answers." Respecting children as authors or artists seems somehow different. Mathematics teachers must respect students' thinking even as they help students to acquire particular tools, concepts, and understandings. Mathematics teachers must respect students' thinking even as they strive to enculturate students into the discourse of mathematics. Hawkins (1972, p. 113) captures some of this tension when he writes that the teacher must be able to "sense when a child's interests and proposals—what I have called his trajectory—are taking him near to mathematically sacred ground. . . . A teacher-diagnosticsian must map a child's question as much as his answer, neither alone will define the trajectory; and he must be prepared to anticipate something of what the child may encounter farther along the path."

Rationale for Teaching Invention

When my students excitedly noticed that only some numbers could be formed into squares out of the ceramic tiles we were using to explore multiplication and division, and that many of the odd numbers yielded only two different rectangles, they were reaching out to square and prime numbers. They were also reaching out to a kind of mathematical thinking: seeing patterns and conjecturing about their generalizability. Riba suggested that there would be more odd numbers that could be made into squares than even numbers; Betsy coun-

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tered, pointing out the even-odd pattern in the squares they had found thus far (1, 4, 9, 16, 25, 36, 49, 64).

When Jeannie and Sheena announced that “you can’t *prove* that an even number plus an odd number would always be an odd number—because numbers go on forever and so you can’t check every one,” the class was shocked:

Mei: (pointing at the “theorems” posted above the chalkboard)
Why did you say *those* were true?

Sheena: She just *thought* of it today.

Ofala: I think that an even plus an odd will always equal an odd because I tried . . . (counting in her notebook) . . . 18 of them and they always came out odd.

Jeannie: But how do you know it will *always* be odd?

(January 26, 1990)

Third graders tread frequently on “mathematically sacred ground” (Hawkins, 1972). They also tread on mathematically uncharted ground. Surely “respecting children’s thinking” in mathematics does not mean ignoring nonstandard insights or unconventional ideas; neither must it mean correcting them. But *hearing* those ideas is challenging. For one thing, teachers are responsible for helping children acquire standard tools and concepts—ideas of mathematical heritage. However, the unusual and novel may consequently be out of earshot. For another, making sense of children’s ideas is not so easy. Children use their own words and their own frames of reference in many ways that are not necessarily congruent with the teacher’s ways of thinking. Both Dewey (1902) and Hawkins (1972) suggest that a teacher’s capacity to hear children is supported by a certain kind of subject-matter knowledge. Hawkins (1974, p. 114) describes it: “A teacher’s grasp of subject matter must extend beyond the conventional image of mathematics. . . . What is at stake is not the . . . end-product that is

usually called mathematics, but . . . the whole domain in which mathematical ideas and procedures germinate, sprout, and take root, and in the end produce visible upper branching, leafing, and flowering.”

So, even when the teacher *hears* the child, what is she supposed to do? What does it mean to respect children’s thinking while working in a specialized domain that has accepted ways of reasoning and working and accepted knowledge (Kitcher, 1984)? Here I explore this problem in the context of one child’s unconventional idea, an idea I chose to extend and develop in class.

Appreciating the Mathematics in the Child

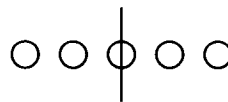
We had been working with patterns with odd and even numbers. One day, as we began class, Sean announced that he had been thinking that six could be both odd *and* even because it was made of “three twos.” Challenged by often-quiet Temba to “prove it to us,” Sean drew on the board:



and explained that since three was an odd number, and there were three *groups*, this showed that six could be both even and odd. At this point, the only explicit definition of even numbers that we had developed held that a number was even “if you can split it in half without having to use halves”:



Six is even because you can split it in half without having to use halves.



Five is not even because you have to split one in half. Five is odd.

Sean had broken from this convention by dividing six into *groups of two* rather than into *two groups*. Although the other children were dubious, they seemed interested.

Mei: I think I know what he is saying . . . is that it's, see. I think what he's saying is that you have three groups of two. And three is an odd number so six can be an odd number *and* an even number.

T: Is that what you are saying, Sean?

Sean: Yeah.

Mei said she disagreed. "It's not according to, like . . . can I show it on the board?"

Pausing for a moment to decide what number to use for her argument, she drew 10 circles and divided them into five groups of two:



Mei: Then why don't you call *other* numbers an odd number and an even number? What about ten? Why don't you call ten an even and an odd number?

Sean: (paused, studying her drawing calmly and carefully) I didn't think of it that way. Thank you for bringing it up, and I agree. I say ten *can* be odd or even.

Mei: (with some agitation) What about *other* numbers? Like, if you keep on going *on* like that and you say that *other* numbers are odd and even, maybe we'll end up with *all* numbers are odd and even! Then it won't make sense that all numbers should be odd and even, because if all numbers were odd and even, we wouldn't be even *having* this discussion!

Ofala, jumping into the fray, was the first to address directly the issue of divisibility by two as including dividing into groups of two. She said she disagreed with Sean be-

cause "if you wanted an odd number, usually, like (and she drew some hash marks on the board), . . . even numbers are something like this. Even numbers have two in them (she circled the hash marks in groups of two) and also *odd* numbers have two in them—except they have *one left*":



I called attention to what I referred to as "Ofala's definition for odd numbers," which she restated as "an odd number is something that has one left over." Her formulation was, I realized, in essence, the formal mathematical definition of an odd number: $2k + 1$. The children tried some experiments with it, with numbers that they expected to work because they already knew them to be odd. Temba tried 3, Betsy tried 21, and Cassandra tried 17. Each time, when they represented the numbers with hash marks and circled groups of two, they found that they had one left over.

Later, Riba was still thinking about what Sean had proposed about some numbers being both even and odd. She said that "it doesn't matter how much circles there are—how much times you circle two, it doesn't prove that six is an odd number." Ofala agreed.

But Sean persisted with this idea that some numbers could be both even and odd. On the one hand, Sean was wrong. Even and odd are defined to be nonoverlapping sets—even numbers being multiples of two and odd numbers being multiples of two plus one. He was, as Riba pointed out, paying attention to something that was irrelevant to the conventional definitions for even and odd numbers—that is, how *many* groups of two an even number has. On the other hand, looking at the fact that six has *three* groups of two and ten has *five* groups of two, Sean noticed that some even numbers have an *odd* number of groups of two. Hence, they were, to him, special. I thought about how I could treat this as a mathe-

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mathematical invention—and whether I should. I wrote in my journal:

I'm wondering if I should introduce to the class the idea that Sean has identified (discovered) a new category of numbers—those that have the property he has noted. We could name them after him. Or maybe this is silly—will just confuse them since it's nonstandard knowledge—i.e., not part of the wider mathematical community's shared knowledge. I have to think about this. It has the potential to enhance what kids are thinking about "definition" and its role, nature, and purpose in mathematical activity and discourse, which, after all, has been a major point this week. What should a definition do? Why is it needed? [Teaching journal, January 19, 1990, pp. 184–185]

I thought about how I want the children to be learning about how mathematical knowledge evolves. I also want them to have experience with what a mathematical community might do when novel ideas are presented. In the end, I decided not to label his claim wrong and, instead, to legitimize Sean's idea of numbers that can be "both even and odd." I pointed out that Sean had invented another kind of number that we had not known before and suggested that we call them "Sean numbers." He was clearly pleased, the others quite interested. I pressed him for the definition of Sean numbers and we got the following: "Sean numbers have an odd number of groups of two." And, over the course of the next few days, some children explored patterns with Sean numbers, just as others were investigating patterns with even and odd numbers. Sean numbers occur every four numbers—why? If you add two Sean numbers, do you get another Sean number?² If a large number ends with a Sean number in the ones place, is the number a Sean number?

Students' Learning

Often I must grapple with whether or not to validate nonstandard ideas. Choosing to legitimize nonstandard *content*—"Sean numbers"—was more difficult than

valuing unconventional *methods*. I worried: Would children be confused? Would "Sean numbers" interfere with the required "conventional" understandings of even and odd numbers? Or would the experience of inventing a category of number, a category that overlaps with others, prepare the children for their subsequent encounters with primes, multiples, and squares? How would their ideas about the role of definition be affected? I was quite uncertain about these questions, but it seemed defensible to give the class firsthand experience in seeing themselves capable of plausible mathematical creations.

When I gave a quiz on odd and even numbers, a quiz that entailed some of the kinds of mathematical reasoning we had been using, the results were reassuring. Everyone was able to give a sound definition of odd numbers and to identify and justify even and odd numbers correctly. And, interestingly, in a problem that involved placing some numbers into a string picture (Venn diagram), no one placed 90 (a Sean number) into the intersection between even and odd numbers. If they were confused about these classifications of number, the quizzes did not reveal it.

Dilemmas of Respecting Children as Thinkers

As a mathematics teacher, I am responsible for certain content. My students are supposed to be able to identify even and odd numbers, add and subtract, measure, understand fractions, and much more. Often my problem is to figure out where they are in their thinking and understanding. Then I must help to build bridges between what they already know and what there is to learn. Sometimes my problem is that it is very difficult to figure out what some students know or believe—either because they cannot put into words what they are thinking or because I cannot track what they are saying. And sometimes, as in this example, students present ideas that are very different from standard mathematics.

The ability to *hear* what children are saying transcends disposition, aural acuity, and knowledge, although it also depends on all of these. And even when you think you have heard, deciding what to do is often a trek over uncharted and uncertain ground. Although Sean was, in a conventional sense, wrong—that is, six is *not* both even and odd—his claim was magnificently at the heart of “doing” mathematics.

Dilemma No. 3: Creating and Using Community

Classrooms as learning communities (Schwab, 1976) are not a new idea. In my teaching, I am trying to model my classroom as a community of mathematical discourse, in which the validity for ideas rests on reason and mathematical argument, rather than on the authority of the teacher or the answer key (cf. Ball, 1988; Lampert, 1986a). In so doing, I aim to develop each individual child’s mathematical power *through the use* of the group. I aim to develop the children’s appreciation for and engagement with others different from themselves. In Schwab’s (1976) terms, we strive to be a learning *community* and also to be *learning community*.

In working our way through alternative ways of approaching and solving problems, we confront issues of shared definition and assumptions, crucial in using mathematics sensibly. I am searching for ways to construct classroom discourse such that the students learn to rely on themselves and on mathematical argument for making mathematical sense. My role in this is tricky: Surely I am the one centrally responsible for their learning the content of the third-grade curriculum. I am also responsible for fostering their capacity and disposition to learn more mathematics and to use it in a variety of life situations. In my work, I often encounter traces of what students learn from their math classes in school and how authority for knowing figures in that learning. For example, one of my students, explaining why she put a little 1 above the

tens column when she was adding, said, puzzled, “That’s what our teacher in second grade *told* us to do whenever you carry.” Never mind that in this case she should have carried a 2 instead of a 1—the underlying principle for the strategy was not the reason, but teacherly authority. Of course, teacherly authority plays a role in my classroom as it does in any classroom. I aim, however, to use my authority to encourage a set of intellectual and social norms to support a kind of work unusual in students’ prior experience in school. Rather than establishing myself as the final arbiter of truth, I strive to develop and distribute in the group a set of shared notions about what makes something true or reasonable. The dilemmas inherent in trying to use the group to advance the individual and vice versa, all while keeping one’s pedagogical eye on the mathematical horizon, are not trivial.

In this section, I use a segment from a lesson on integers, on one of the days we were struggling with the building model (introduced previously under dilemma no. 1) and trying to make sense as a community of mathematical thinkers. The vignette spotlights the dilemmas of my role, of authority for knowledge, and of the clarifying-confusing tensions inherent in group discussions—all critical aspects of creating and maintaining a community.

Community Learning

The students were stuck on a problem involving negative numbers. What could it mean to try to do $6 + (-6)$? What could be the answer? All of them were convinced that $-6 + 6 = 0$. This was established by use of Nathan’s conjecture (which was actually a theorem, but had not been yet labeled as such): “Any number below zero plus that same number above zero equals zero.” It was a little surprising to me that no one put this together with the commutativity of addition to argue that, if $-6 + 6 = 0$, then $6 + (-6)$ would have to equal zero as well. That not one child made this

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connection was striking and reminded me of the shifts we assume in conventional mathematics teaching. When children are introduced to rational numbers, for instance, they are simply supposed to carry their notions about operations with them into this new domain.

Perhaps I might have chosen at this point to pose a challenge: "What if someone in the other third-grade class came over and said, 'Nathan's conjecture says that any number below zero, plus that same number above zero equals zero and I think you could turn it around because $3 + 6$ is the same as $6 + 3$, so you can turn Nathan's conjecture around too, and so I think that the answer to $6 + (-6)$ is 0?' What would you say?" This is one strategy I use when the group has entrenched itself in an inadequate or incorrect conclusion or assumption. I did not do this in this case, however. It seemed to me that the students were right not to assume that what they knew for positive numbers would automatically hold for negatives. Still, you ask, why not press them a bit? It seemed to me a big step to figure out and reason about the arithmetic of integers, and I wanted to let it simmer for a while. I thought, too, I could construct an alternative representation with which they could figure out what made sense.

Recall the children's struggles over this problem of $6 + (-6)$. Sean had argued that $6 + (-6)$ should just be six "because it wouldn't be able to *do* anything. It just stays the same, it stays on the same number. Nothing is happening." And Betsy had, intuitively, put two little paper people on the drawing of the building and moved them toward each other until they met—at zero. In both cases, I remained silent, not presenting the children with questions to challenge their solutions. I might have asked Sean, as Riba did, "It says plus six below zero. You're supposed to do *something*. You can't just leave it alone." Or, I might have pressed Betsy, whose conclusion was right but whose reasoning incomplete, "What

would you do if it said $6 + (-2)$? Or, "Why don't you put two people on the building and move them toward each other when you add two numbers *above* zero—like $6 + 6$?" Instead, however, the other children pressed them:

- Betsy: Instead of Sean's, I got zero.
 Teacher
 (T): You'd like to put zero here for $6 + (-6)$?
 Betsy: Do you want to see how I do it?
 T: Okay.
 Other students: Yeah!
 Betsy: Here. You're here, but you can't go up to twelve, because that's six plus *six*. So, I say it's just the opposite. It's just six minus six.
 Sean: But it says plus, not minus!!!
 Betsy: But, you're minusing.
 Riba: Where'd you get the minus?
 Sean: You should just leave it alone. You can't *add* six below zero, so you just leave it. Just say "good-bye" and leave it alone and it is still just six.
 Mei: But this six below zero would just disappear into thin air!
 Sean: I know. It would just disappear because it wouldn't be able to *do* anything. It just stays the same, it stays on the same number. Nothing is happening.
 Betsy: But, Sean, what would you do with this six below zero then?
 Sean: You just say "good-bye" and leave it alone.
 Riba: You can't *do* that. It's a *number*.
 Sean: I know, but it's not going down. It's going up because it says plus.
 Mei: I think I disagree with Betsy and Sean because I came up with the answer nine.
 T: Okay, why don't you come and show us how you did that. (At this point I did not have a clue what Mei was thinking.)
 Mei: (reaches up and places one of the little paper people on the building) I start here (at 6 above zero) and then I add three to that, because when

you go three and three—it's six. Yeah, and then I got nine, so I think the answer is nine.

T: Lucy?

Lucy: Where did the other three go then?

Mei: Well, see 'cause it's three below zero. . . .

Sheena: I know what you're saying.

Mei: So when we put two in each group in order to make one because it's below zero.

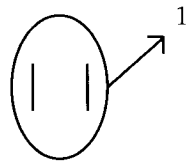
(I still had no idea what she was doing, but I assumed that if she explained it further, it would make sense in some way.)

T: I don't understand this part—put two in each group in order to make 1.

Mei: If we take six and add six to it, we get twelve above zero, but it's below zero, so—and three plus three is six, so we add three more to the six above zero.

Riba: Mei, is this what you're saying? Three and three makes six. And then you're saying six below, and since it's *below*, you have to go up to the three?

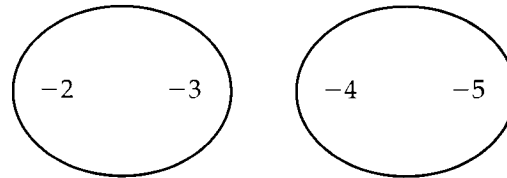
Mei: I'm making, this is one of the numbers, these are two of the numbers below zero (she made two hash marks to represent two of the numbers), and two of these equals 1 (she wrote 1), and if I have about, like—



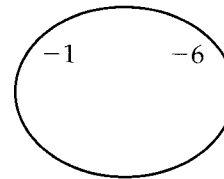
Betsy: What numbers are these? Can you put the numbers in yet?

Mei: (She paused, wrinkling her face and pondering this request,

and then drew the following on the board):



Okay, let's see. Two below zero and three below zero, and this could be four below zero and five below zero, equals 1.



And then I have two already, and that means you will go up to eight. (She moved the paper person from the 6 to the 8 on the building.) And then I make one more. One below zero and six below zero, so there's one more then to go up, and now I end up on nine. (She moved the person one more floor up.)

When she put in numbers at Betsy's request, I realized that Mei had not been thinking of particular numbers. She had meant any two numbers below zero would equal 1 (see her first drawing with hash marks) and that you could make three pairs of "below zero" numbers because the problem said " -6 ." I think Mei was working off a memorized "fact" that "a negative plus a negative equals a positive," something she may have been told by some helpful person. So, taking six below zero and pairing the six into three groups of negative numbers (again, look at her drawing with the hash marks), you would get three positive and would add that three to the six above zero—hence, the answer 9.

Sean: I don't understand what you're trying to say. I thought that

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you were starting from the six, plus six below, not like 1 plus 1 below zero plus six or any other. You're doing all different numbers.

The discussion continued for about 10 more minutes. Ofala said she didn't agree with either Betsy or Mei "because it says plus, and you are supposed to be going *up*." Mei replied that if you go up, you end up on the twelfth floor, and that is the answer for $6 + 6$, not $6 + (-6)$. This made sense to Ofala, who then revised her answer. (We use the term "revise" to denote "changing one's mind," in place of more traditional notions of correcting or fixing or being wrong.) Other children spoke up, either agreeing with one of the presented solutions or questioning one, for example, "If you're going to start with 6, then you have to go up because it's plus?" Sheena objected, "So you're saying that six plus six equals 12 and six plus six below also equals 12? I don't get it."

Jeannie, who had been quiet all this time, raised her hand. "Jeannie?" I asked. "I'm *confused*," she began slowly. "Betsy said that it is zero, and Mei says that it is nine, and Ofala says that it is twelve, and Sean says that it is six, and I don't know who to believe." I asked her what she had thought when she worked on it before we started the discussion. She said she thought it was zero (the correct answer), "but now I'm not sure."

At this, Cassandra raised her hand. She had changed her mind, listening to the discussion. "I get my person and I started at six and I went down six more and I ended up at zero." Although this was the end of class and came on the heels of Jeannie's confusion, I still refrained from sealing the issue with my approval. I asked Cassandra why she thought she should go down. "Because it says below zero." Cassandra was now getting the right answer, but her reason was problematic. For instance, when she tries to subtract a negative number someday, "going down" will be wrong.

This is a problem that arises regularly: children say things that are true in their current frame of reference, in relation to what they currently know, but that will be wrong in other contexts later on. (An example may help here. When a first grader announces that 3 is the next number after 2, he is right—in his domain, which is the counting numbers. But, for a sixth grader considering rational numbers, there is no next number after 2, for the rational numbers are "infinitely dense," which means that between any two rational numbers, there is another rational number. Between 2 and 2.1 are 2.01, 2.02, and so on. Between 2 and 2.01 are 2.001, 2.002, and so on. Consequently, there is no "next number" unless you specify a context—e.g., the next hundredth.) I chose *not* to correct Cassandra's statement, believing that my qualification of what she said would just pass the children by anyway. But, as I always do when this happens, when I leave a problematic assertion or answer alone, I felt a sense of unease and dishonesty.

I knew that some others probably felt as confused as Jeannie did at that moment. She seemed matter-of-fact about her confusion, rather than distressed. Still, she was confused. And here we were, at the end of the class period. I glanced at the clock and saw that we had 15 minutes, and I made a decision. Moving up by the board, I announced,

I want everybody to stop talking for 1 minute now, just think for a minute. I'd like you to find in your notebook where there's an empty space right now. I want you to write down two things. Listen very carefully because you'll have 5 minutes to do this and I want you to do it carefully. The first thing I want you to write down is what the argument was about that we've been having today, what are we trying to figure out? And then I want you to write down who you agree with most, or if you don't agree with anybody who's up on the board right now, write what you think about this argument. You might not be sure, but write down what you think right now as

of October 12, on Thursday at 1:50 p.m. The first thing is, what do you think we've been arguing about, and the second thing is, what do you think about the argument? What do you think the answer should be? And why you think that. There's the first question, what have we been arguing about? The second one is, what do you think and why? [October 12, 1989]

The room was silent as the children wrote intently in their notebooks. Ten out of the 17 children who were in class that day agreed with Betsy, who had argued that $6 + (-6) = 0$ (the correct answer). Riba said that "Betsy's ikspachan [explanation]" caused her to change her mind. Two agreed with Sean that the answer should be six. Sheena wrote that she disagreed with Betsy: "betsy is using a minece instead of a plus and its says plus not minece." Three students were not sure. Jeannie said she wasn't "srue hoo to balve [sure whom to believe]," although she soon thereafter became convinced that the answer was zero.

Students' Learning

We continued to struggle for the next few days with making sense of adding and subtracting negative numbers. I tried to think of better representations for exploring this. When we moved on from negative numbers a week or so later almost every student was able to add and subtract integers accurately if the negative number was in the first position, for example, $-5 + 4$, or $-3 - 8$. And many who relied on commutativity or money were able to operate readily with addition and subtraction sentences in any form. This was not a bad achievement.

In addition to learning specifically about operations with integers, what might the students have been learning about community or about the roles of different people—their peers, the teacher, themselves—in their learning? Evidence on this is harder to obtain, but a few snatches from other points, later in the year, help to illuminate some possible learnings. One day, after we

had had a particularly long and confusing session on even and odd numbers, I asked the students for comments on the discussion (January 19, 1990). Sheena commented that "it helps" to hear other people's ideas because "it helps you to understand a little bit more." She gave an example: "I didn't think zero was even or odd until yesterday and then someone said it could be even because one below zero and one above zero are both odd, and that made sense."

Mei made a comment that was reminiscent of Jeannie's confusion over the $6 + (-6)$ discussion: "I thought zero was an even number, but from the meeting [the discussion] I got sort of mixed up because I heard other ideas I agree with and now I don't know which one I should agree with." Once again, I saw that children were becoming confused *from* the discussions. But then I asked Mei what she was going to do about this. What she said was significant for what it revealed about what she may have been coming to understand about herself and about learning mathematics: "I'm going to listen more to the discussion and find out." Both Sheena and Mei, like many of their peers, seemed by midyear to have the sense that they could figure things out together, in group discussions, as well as alone.

I asked the class how they felt when, during a discussion, they were arguing a position with which many other people disagreed. Jeannie said that it did not bother her: "I don't really care how *many* people think [something]. If they changed my mind—if they convince me, then I would change my mind" (January 19, 1990). I asked how they felt when they took a position that no one else in the class was taking. Sean said he "felt fine" about that and that he, too, changed his mind when he was convinced: "I have just changed my mind about 1—that it *is* an odd number." Some children, however, have complained that some of their classmates argue *too* much and that the discussions go on for a long

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time and “we never find answers” (January 19, 1990).

In general, though, the students seemed to be developing a sense for what they could learn from one another. Riba commented that discussions are helpful because one person may have a good idea when it is taking a long time to figure it out all by yourself. Mei added that, in discussions, “we get ideas from other people.” And Sheena said that “it helps us to learn what other people’s thoughts are about math because they might teach something new that you never knew before.” Or give us a good example, added Ofala. Even, said Riba, maybe the whole class would agree that something was right and only one person in the class would be able to prove that it was wrong.

Dilemmas of Creating and Using Community

Despite evidence that the third graders learn to learn on their own as well as from one another, there are many days on which I ask myself whether this is time well spent. Take the discussion of $6 + (-6)$, for example. We spent over half an hour discussing what would be a sensible answer for that one problem. The correct answer was given, but with a problematic explanation. Moreover, two other answers were presented and given equal discussion time. I did not tell or lead the students to conclude that $6 + (-6)$ equals zero—by pointing them at the commutativity of addition or at the need for the system of operations on integers to be sensibly consistent. At the end of class, only slightly over half the students knew the right answer. And some misconceptions were floating around—that any negative number plus another negative number equals one positive, for example. Still, the very fact that Mei had carried this misconception into class—probably based on something someone had explained to her about subtracting a negative number—is the kind of thing that keeps me thinking that time spent unpacking ideas is time val-

uably spent. I have too often been confronted with evidence of what students fail to understand and fail to learn from teaching that strives to fill them efficiently with rules and tools. It is not clear to me that *telling* them that $6 + (-6) = 0$ will result in more enduring or resilient understanding, or in better outcomes in terms of what the children *believe* they are capable of learning.

Two issues lie at the heart of creating and using community in a third-grade mathematics classroom: one centered on my role and authority for knowing and learning mathematics, and another on balancing confusion and complacency in learning. These two issues are intertwined all of the time: How much should I let the students flounder? Just because it took hundreds of years for mathematicians to accept negative numbers does not necessarily imply that third graders must also struggle endlessly with incorporating them into their mathematical domain. How much “stuckness” is productive to motivate the problems that are being pursued? Deciding when to provide an explanation, when to model, when to ask rather pointed questions that can shape the direction of the discourse—is delicate and uncertain. Certainly mathematical conventions are not matters for discovery or reinvention—for instance, how we record numbers or what a square is. But that $6 + (-6)$ must equal zero, or that an even number plus an odd number will always be odd, or that the probability of rolling a seven with two standard dice is $6/36$ are things that children *can*—through conjecture, exploration, and discussion—create. Children can also create—as Sean did—new mathematics, new beyond its novelty only for third graders. When is this important?

As the teacher, I know more mathematics than my third graders. There is a lot of mathematics for them to learn. If I understand that $6 + (-6)$ equals zero and can explain it clearly, it may make sense for me to show them how you add a negative num-

ber, and get on with more important things. Yet, orchestrating a classroom community in which participants work together to make sense, developing strategies and ideas for solving mathematical and real-world problems, implies a set of goals that do not exclude, but are not limited to, the children's developing understandings of operations on integers.

The classroom community is often, as the children themselves note, a source of mathematical insights and knowledge. The students hear one another's ideas and have opportunities to articulate and refine or revise their own. Their confidence in themselves as mathematical knowers is often enhanced through this discourse. Still, as the story about $6 + (-6)$ shows, the community can also be a stimulus for confusion. Students with right answers become unsettled in listening to the discussion and sometimes end class uncertain and confused. Are their apparently fragile understandings best strengthened by exposing them to alternative arguments? I worry and I wonder about providing more closure: I as often open or conclude class discussions with a summary of what our open problems, conjectures, and puzzlements are as I do with a summary of what we have learned. Are the students learning from this slow progress to tentative conclusions that anything goes, that there are no right answers? Or are they learning as I would like them to, that understanding and sensible conclusions often do not come without work and some frustration and pain—but that they can do it, and that it can be immensely satisfying?

Dilemmas of Trying to Be "Intellectually Honest" in Teaching Mathematics

In what sense is my practice with third graders "intellectually honest" (Bruner, 1960)? It is honest in its frame—in my concern for students' opportunities to learn about mathematical content, discourse, and community. I try to focus on significant mathematical content and I seek to fashion

fruitful representational contexts for students to explore. To do this productively, I must understand the specific mathematical content and its uses, bases, and history, as well as be actively ready to learn more about it through the eyes and experiences of my students. My practice is also honest in its respect for third graders as mathematical thinkers. In order to generate or adapt representations, I must understand a lot about 9-year-olds: What will make sense to them? What will be interesting? How will they take hold of and transform different situations or models? I must consider the mathematics in relation to the children and the children in relation to the mathematics. My ears and eyes must search the world around us, the discipline of mathematics, and the world of the child with both mathematical and child filters. And from all of these aims and principles come the dilemmas that lie at the core of creating a defensible practice: If children believe that zero is not a number, and they are all convinced and agree, what is my role? If all the fraction models I can think of still mislead and distort in some ways, what should I do? When students construct a viable idea that is, from a standard mathematical perspective, reasonable but incorrect, how should I respond?

Dilemmas such as these are not solely the product of the current educational reform rhetoric; many are endemic to teaching (Lampert, 1985). Practice is, after all, inherently uncertain (Jackson, 1986; Lortie, 1975). Still, aiming to create a practice that is, at once, honest to mathematics and honoring of children clearly heightens the uncertainties. The conception of content is more uncertain than a traditional view of mathematics as skills and rules, the view of children as thinkers more unpredictable. Lampert (1985) argues, however, that embracing—rather than trying to resolve—pedagogical dilemmas gives teachers a power to shape the course and outcomes of their work with students. My understandings and assumptions about 9-year-olds

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equipped me to make decisions about mathematical representation and activity that served their opportunities to learn. Similarly, my notions about mathematics allowed me to hear in the students' ideas the overtures to important understandings and insights.

Because no rules can specify how to manage and balance among competing concerns, teachers must be able to consider multiple perspectives and arguments and to make specific and justifiable decisions about what to do (Lampert, 1986b). Teachers need "the resources to cope with equally weighted alternatives when it is not appropriate to express a preference between them"; they need to be comfortable with "a self that is complicated and sometimes inconsistent" (Lampert, 1985, p. 193). We need to learn more about what are the crucial resources for managing the dilemmas of mathematical pedagogy.

Like many others, I have assumed that teachers who understand subject matter deeply are better equipped to help students learn with understanding a mathematics that has both personal and disciplinary integrity and worth. And, as a teacher educator, I have worried about the problem of helping teachers transcend their own school experiences with mathematics in order to create new practices of mathematical pedagogy. That mathematical knowledge is helpful is obvious; the kind and quality of such knowledge are less clear. The same is true for knowledge about students and about learning. Although learning mathematics has often—at least in the United States—been considered an exclusively psychological matter, other perspectives—linguistic, cultural, sociological, historical—are equally helpful in learning to listen to and interact with children as learners. And I am increasingly aware that there are many resources beyond knowledge that contribute to wise practice: patience, respect, flexibility, humor, imagination, and courage, for instance.

In a society in which mathematical success is valued and valuable, reforms that herald a richer understanding and power for students are attractive. But the pedagogical courses are uncertain and complex. How teachers learn to frame and manage the dilemmas of "intellectually honest" practice in ways that do indeed benefit all students is crucial to the promise of such work. (See, e.g., Lampert, *in press*; Lensmire, 1991; Wilson, *in press*, all of whom write about the special dilemmas they have found in their efforts to construct alternative pedagogies in mathematics [Lampert], writing [Lensmire], and social studies [Wilson].)

In the face of these kinds of challenges, attention to and debates about what teachers need to know—while important—seem insufficient. Another resource worthy of development is the professional community of teachers and the discourse about practice in which teachers might engage. Typically teachers face the problems and dilemmas of their work alone. Isolated from one another, rarely do they have satisfying or helpful opportunities to talk about practice. To begin with, the structure of teachers' work mitigates against these kinds of opportunities. Furthermore, the incentives for honest and constructive conversation are lacking. On the one hand, acknowledging pedagogical difficulty is too often tantamount to admitting professional incompetence. On the other hand, the tone of some articles and workshops seems to convey that there is "a right way" to motivate children, to teach place value, or to respond to certain kinds of questions from students. Between these two opposing approaches to problems of practice lies little territory for thoughtful teachers to discuss with others the uncertain challenges of their work.

Representing content, respecting students, creating and using community—these are not aims simply resolved. However, that these aims present tough challenges does not mean that all efforts are equal. That all representations have limitations does not mean that any represen-

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tation is as good as any other. Not every claim to respect students stands up to critical scrutiny. And not all applications of the concept of community are equally defensible. Developing shared standards for evaluating pedagogical interpretations and actions is a necessary step toward restructuring the social and intellectual parameters of teachers' work. Recently I experimented with using videotape from my classroom to engage others in thinking about my dilemma with Sean's numbers. I am still unsure about the choices I made across those few days in my class, and it felt risky to open up my practice to the scrutiny of strangers. I thought people might scoff at my labeling this a dilemma. I thought others might just tell what I should have done—which, while possibly helpful, would underestimate the complexity of my dilemma. The videotape did prove to be a fruitful context for discussion: I watched and listened to alternative interpretations and ideas about that lesson, about what Sean was thinking, about what the other students were doing. My own thinking about how I responded was expanded, as was the thinking of the other participants in the conversations. I saw the promise of a kind of professional discourse that does not expect single answers to complicated problems of practice, but which, instead, proffers tools for interpretation and choice. Such tools, gleaned in forums for professional exchange, could become an important resource for improving teaching and learning in ways that are both responsive to students and responsible to content. Dilemmas endemic to this kind of teaching could be identified and explored. The important terrain between "it's all a matter of individual style," on the one hand, and five-step models of instruction, on the other, could be developed. What makes the building model a useful representation—and for what purposes? What makes it a troubling one? What are the arguments for and against following Sean's assertion that six is both even and odd? This requires devel-

oping a kind of principled discussion that has been all too rare in either research or practice.

Although such discussion would require changes in the structure of teachers' work, structural change is not all that is needed. These kinds of exchanges would require substantial revisions in the norms of professional discourse—in what educators (teachers and others) talk with one another about and in what ways. This would require rethinking assumptions about what counts as evidence for believing or doing something in teaching. And such exchanges would require teachers letting one another and others behind the proverbial classroom door, to explore one another's practices, to raise hard questions, and to help one another grow.

Notes

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¹ All names used are pseudonyms and are drawn appropriately, to the extent possible, from the individual children's actual linguistic and ethnic backgrounds. They also accurately reflect the students' gender.

² Because a Sean number has an odd number of groups of two, the sum of two Sean numbers will have an *even* number of groups of two (because odd + odd = even) and so will *never* equal a Sean number.

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